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2006 J. Phys. A: Math. Gen. 39 12153

(http://iopscience.iop.org/0305-4470/39/39/S09)

Institute of Physics Publishing Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **39** (2006) 12153–12165 doi:10.1088/0305-4470/39/39/S09

A family of solutions of a higher order PVI equation near a regular singularity

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Received 16 January 2006
Published 13 September 2006
Online at stacks.iop.org/JPhysA/39/12153

Abstract

Restriction of the N-dimensional Garnier system to a complex line yields a system of second-order nonlinear differential equations, which may be regarded as a higher order version of the sixth Painlevé equation. Near a regular singularity of the system, we present a $2N$-parameter family of solutions expanded into convergent series. These solutions are constructed by iteration, and their convergence is proved by using a kind of majorant series. For simplicity, we describe the proof in the case $N = 2$.

PACS number: 02.30.Hq
Mathematics Subject Classification: 34M55, 34M35

1. Introduction

Let us consider a Fuchsian differential equation of the form

$$\frac{1}{y}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sum_{i=1}^{N+2} \frac{c_i}{(x-t_i)^2} + \frac{c_{N+3}}{x(x-1)} + \sum_{i=1}^{N} \frac{A_i}{x(x-1)(x-t_i)} + \sum_{j=1}^{N} \left(\frac{3}{4(x-\lambda_j)^2} + \frac{B_j}{x(x-1)(x-\lambda_j)} \right)$$

with the regular singularities $x = t_1, \ldots, t_N, t_{N+1} := 0, t_{N+2} := 1, t_{N+3} := \infty$ and the non-logarithmic singularities $x = \lambda_1, \ldots, \lambda_N$. The isomonodromic deformation with respect to the parameters $t_1, \ldots, t_N$ yields the N-dimensional Garnier system

$$\frac{T'(t_i)(t_i-\lambda_j)}{\Lambda(t_i)}\frac{\partial \lambda_j}{\partial t_i} - \frac{T'(t_k)(t_k-\lambda_j)}{\Lambda(t_k)}\frac{\partial \lambda_j}{\partial t_k} = \frac{(t_i-t_k)T(\lambda_j)}{(\lambda_j-t_i)(\lambda_j-t_k)\Lambda'(\lambda_j)} \qquad (k \neq i),$$

$$\frac{\partial^2 \lambda_j}{\partial t_i^2} = \frac{1}{2}\left(\frac{T'(\lambda_j)}{T(\lambda_j)} - \frac{\Lambda''(\lambda_j)}{2\Lambda'(\lambda_j)}\right)\left(\frac{\partial \lambda_j}{\partial t_i}\right)^2 - \left(\frac{T''(t_i)}{2T'(t_i)} - \frac{\Lambda'(t_i)}{\Lambda(t_i)}\right)\frac{\partial \lambda_j}{\partial t_i}$$

$$+\frac{1}{2}\sum_{\substack{l=1\\l\neq j}}^{N}\frac{T(\lambda_j)\Lambda'(\lambda_l)(\lambda_l-t_i)^2}{T(\lambda_l)\Lambda'(\lambda_j)(\lambda_j-t_i)^2(\lambda_j-\lambda_l)}\left(\frac{\partial\lambda_l}{\partial t_i}\right)^2$$
$$-\sum_{\substack{l=1\\l\neq j}}^{N}\frac{\lambda_j-t_i}{(\lambda_l-t_i)(\lambda_l-\lambda_j)}\frac{\partial\lambda_j}{\partial t_i}\frac{\partial\lambda_l}{\partial t_i}+\frac{2\Lambda(t_i)^2T(\lambda_j)}{T'(t_i)^2(\lambda_j-t_i)^2\Lambda'(\lambda_j)}$$
$$\times\left(\sum_{k=1}^{N+3}(c_k+3/4)-2+\sum_{\substack{k=1\\k\neq i}}^{N+2}\frac{(c_k+1/4)T'(t_k)}{\Lambda(t_k)(\lambda_j-t_k)}+\frac{c_iT'(t_i)}{\Lambda(t_i)(\lambda_j-t_i)}\right),$$

$i, j = 1, \ldots, N$, with $T(x) = \prod_{i=1}^{N+2}(x-t_i)$, $\Lambda(x) = \prod_{j=1}^{N}(x-\lambda_j)$; which has fixed singularities along the hyperplanes $t_i = t_j$ $(1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N+3;\ i \neq j)$ ([4, 5, 8, 13, 14]). The unknown vector function $(Q_1, \ldots, Q_N)$, $Q_j = t_j\Lambda(t_j)T'(t_j)^{-1}$, whose entries are essentially elementary symmetric functions of $\lambda_1, \ldots, \lambda_N$, satisfies another system of equations corresponding to polynomial Hamiltonian structure ([13]). When $N = 1$, as was derived by Fuchs ([2, 3]), this system coincides with the sixth Painlevé equation

$$\mathrm{PVI:}\quad \lambda''=\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)(\lambda')^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\lambda'$$
$$+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(b_\infty-\frac{(c_2+1/4)t}{\lambda^2}+\frac{(c_3+1/4)(t-1)}{(\lambda-1)^2}-\frac{c_1t(t-1)}{(\lambda-t)^2}\right),$$

$b_\infty = \sum_{k=1}^{4} c_k + 1$ $(\lambda := \lambda_1, t := t_1,\ ' = \mathrm{d}/\mathrm{d}t)$. Restriction of the Garnier system with $N \geqslant 2$ to the complex line $(t_2, \ldots, t_N) = (s_{0,2}, \ldots, s_{0,N})$ $(s_{0,i} \in \mathbb{C}\setminus\{0,1\},\ s_{0,i} \neq s_{0,\iota}\ (i \neq \iota))$ yields an N-dimensional system of second-order nonlinear equations, which may be regarded as a higher order version of PVI. Putting

$$t := t_1, \qquad \boldsymbol{\lambda}_{\wedge j} := (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_N),$$

we write this system in the form

$$\lambda_j'' = \Phi_N(t, \lambda_j, \boldsymbol{\lambda}_{\wedge j}, \lambda_j', \boldsymbol{\lambda}_{\wedge j}'), \qquad j = 1, \ldots, N \tag{1.1}$$

$(' = \mathrm{d}/\mathrm{d}t)$. Here $\Phi_N(t, \lambda, \boldsymbol{\mu}, \tilde\lambda, \tilde{\boldsymbol{\mu}})$ is a rational function of $(t, \lambda, \mu_2, \ldots, \mu_N, \tilde\lambda, \tilde\mu_2, \ldots, \tilde\mu_N)$ obtained from the right-hand member of the second equation in the Garnier system (with $i = 1$) after the substitution

$$(t_1, t_2, \ldots, t_N) \mapsto (t, s_{0,2}, \ldots, s_{0,N}),$$
$$(\lambda_j, \boldsymbol{\lambda}_{\wedge j}) \mapsto (\lambda, \boldsymbol{\mu}), \qquad (\partial\lambda_j/\partial t_1, \partial\boldsymbol{\lambda}_{\wedge j}/\partial t_1) \mapsto (\tilde\lambda, \tilde{\boldsymbol{\mu}}).$$

Note that $\Phi_N(t, \lambda, \boldsymbol{\mu}, \tilde\lambda, \tilde{\boldsymbol{\mu}})$ is independent of j. For example, $\Phi_2(t, \lambda, \mu, \tilde\lambda, \tilde\mu)$ is written in the form

$$\Phi_2(t,\lambda,\mu,\tilde\lambda,\tilde\mu)=\frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}+\frac{1}{\lambda-s_0}-\frac{1}{\lambda-\mu}\right)(\tilde\lambda)^2$$
$$-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{t-s_0}-\frac{1}{t-\lambda}-\frac{1}{t-\mu}\right)\tilde\lambda$$
$$+\frac{\lambda(\lambda-1)(\lambda-s_0)(\mu-t)}{2\mu(\mu-1)(\mu-s_0)(\lambda-t)(\mu-\lambda)}(\tilde\mu)^2-\frac{\lambda-t}{(\mu-t)(\mu-\lambda)}\tilde\lambda\tilde\mu$$
$$+\frac{2\lambda(\lambda-1)(\lambda-t)(\lambda-s_0)(\mu-t)^2}{t^2(t-1)^2(t-s_0)^2(\lambda-\mu)}$$

$$\times \left(a_\infty - \frac{a_0 s_0 t}{\mu\lambda^2} + \frac{a_1(s_0-1)(t-1)}{(\mu-1)(\lambda-1)^2} + \frac{a_2 t(t-1)(t-s_0)}{(\mu-t)(\lambda-t)^2} + \frac{a_3 s_0(s_0-1)(s_0-t)}{(\mu-s_0)(\lambda-s_0)^2} \right), \tag{1.2}$$

$$a_\infty := \sum_{i=1}^{5} c_i + \frac{7}{4}, \qquad a_0 := c_3 + \frac{1}{4}, \qquad a_1 := c_4 + \frac{1}{4},$$

$$a_2 := c_1, \qquad a_3 := c_2 + \frac{1}{4},$$

where $s_0 := s_{0,2}$. By the symmetry of the Garnier system, as far as local properties of (1.1) near fixed singularities are concerned with, it is sufficient to examine near $t = 0$ ([8, 11]). Near $t = 0$, PVI admits a general solution expressed by a convergent series; it satisfies $\lambda(t) = \mathrm{e}^{-\kappa} t^{\omega}(1 + O(|t| + |\mathrm{e}^{-\kappa} t^{\omega}| + |\mathrm{e}^{\kappa} t^{1-\omega}|))$ as $t \to 0$ through a certain domain in the universal covering of $\mathbb{C}\backslash\{0\}$, where $\omega \in \mathbb{C}\backslash(\{\omega \in \mathbb{R} \mid \omega \leqslant 0\} \cup \{\omega \in \mathbb{R} \mid \omega \geqslant 1\})$ and $\kappa \in \mathbb{C}$ are integration constants ([15–17], see also [10, 18]). Furthermore, connection problems are studied through isomonodromic deformation ([1, 6, 7, 9]). For a Hamiltonian system associated with the N-dimensional Garnier system, Kimura *et al* ([12]) gave a reduction theorem around its regular singularity, which may yield a convergent series expression of solutions of (1.1) near $t = 0$, under certain conditions on c_1, c_{N+1} and integration constants.

In this paper, applying the method developed in [17] directly to system (1.1), we present a family of solutions near $t = 0$ expanded into a different kind of convergent series whose coefficients are rational functions of integration constants; and they are valid for generic values of integration constants without any condition on c_i. For a technical reason, we treat (1.1) for examining the behaviour of solutions near the regular singularity, from which a result on the system of $(Q_1, \ldots, Q_N)$ immediately follows. Our main result is stated as follows:

Theorem 1.1. *Suppose that* $s_{0,i} \neq 0, 1$ *for* $2 \leqslant i \leqslant N$, *and that* $s_{0,i} \neq s_{0,\iota}$ *for* $\iota \neq i$. *Let* $\Omega \subset \mathbb{C}$, $\mathbf{K} \subset \mathbb{C}^{N-1}$ *and* $\mathbf{M} \subset \mathbb{C}^{N-1}$ *be arbitrary bounded domains satisfying*

$$\mathrm{cl}(\Omega) \subset \Omega_0 := \mathbb{C} \setminus (\{\omega \in \mathbb{R} \mid \omega \leqslant 0\} \cup \{\omega \in \mathbb{R} \mid \omega \geqslant 1\})$$

and

$$\mathrm{cl}(\mathbf{M}) \subset \mathbf{M}_0 := \mathbb{C}^{N-1} \setminus \left(\bigcup_{i=2}^{N} \bigcup_{\iota=2}^{N} S_{i\iota}^{(0)} \right) \setminus \left(\bigcup_{i=2}^{N} \bigcup_{\iota=i+1}^{N+2} S_{i\iota}^{(1)} \right),$$

where $\mathrm{cl}(\cdot)$ *denotes the closure of each domain, and* $S_{i\iota}^{(\cdot)}$ *are hyperplanes in the* $(\zeta_2, \ldots, \zeta_N)$-*space defined by*

$$S_{i\iota}^{(0)} : \zeta_i = s_{0,\iota} \qquad (2 \leqslant \iota \leqslant N),$$
$$S_{i\iota}^{(1)} : \zeta_i = \zeta_\iota \qquad (i+1 \leqslant \iota \leqslant N),$$
$$S_{i,N+1}^{(1)} : \zeta_i = 0, \qquad S_{i,N+2}^{(1)} : \zeta_i = 1.$$

Denote by $\mathcal{R}_0$ *the universal covering of* $\mathbb{C} \setminus \{0\}$. *Then, for a sufficiently small positive number* $r_0 = r_0(\Omega, \mathbf{K}, \mathbf{M})$, *system (1.1) admits a* $2N$-*parameter family of solutions*

$$\lambda_j = \lambda_j(\omega, \kappa_0, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0; t) \qquad (j = 1, \ldots, N),$$
$$\boldsymbol{\kappa}_1 := (\kappa_{1,2}, \ldots, \kappa_{1,N}), \qquad \boldsymbol{\mu}_0 := (\mu_{0,2}, \ldots, \mu_{0,N}),$$
$$(\omega, \kappa_0, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) \in \Omega \times \mathbb{C} \times \mathbf{K} \times \mathbf{M}$$

whose series expansions

$$\lambda_1(\omega, \kappa_0, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0; t) = \mathrm{e}^{-\kappa_0} t^{\omega} \Bigg(1 + \sum_{p \geqslant 1} \alpha^0_p(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) t^p + \sum_{\substack{p \geqslant 0 \\ q \geqslant 1}} \alpha^1_{pq}(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) t^p (\mathrm{e}^{-\kappa_0} t^{\omega})^q + \sum_{\substack{p \geqslant 0 \\ q \geqslant 1}} \alpha^2_{pq}(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) t^p (\mathrm{e}^{\kappa_0} t^{1-\omega})^q \Bigg),$$

$$\lambda_l(\omega, \kappa_0, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0; t) = \mu_{0,l} + \mathrm{e}^{-\kappa_0} t^{\omega} \Bigg(\kappa_{1,l} + \sum_{p \geqslant 1} \beta^0_p(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) t^p + \sum_{\substack{p \geqslant 0 \\ q \geqslant 1}} \beta^1_{pq}(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) t^p (\mathrm{e}^{-\kappa_0} t^{\omega})^q + \sum_{\substack{p \geqslant 0 \\ q \geqslant 1}} \beta^2_{pq}(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) t^p (\mathrm{e}^{\kappa_0} t^{1-\omega})^q \Bigg) \quad (2 \leqslant l \leqslant N)$$

converge absolutely and uniformly in the domain

$$\Delta_0(\Omega, \mathbf{K}, \mathbf{M}, r_0) := \big\{ (\omega, \kappa_0, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0, t) \in \Omega \times \mathbb{C} \times \mathbf{K} \times \mathbf{M} \times \mathcal{R}_0 \mid |t| < r_0, |\mathrm{e}^{-\kappa_0} t^{\omega}| < r_0^{1/2}, |\mathrm{e}^{\kappa_0} t^{1-\omega}| < r_0^{1/2} \big\},$$

where $\alpha^0_p, \alpha^1_{pq}, \alpha^2_{pq}, \beta^0_p, \beta^1_{pq}, \beta^2_{pq} \in \mathbb{C}(\omega, \boldsymbol{\mu}_0)[\boldsymbol{\kappa}_1]$, *namely, they are polynomials in* $\boldsymbol{\kappa}_1$ *whose coefficients are rational functions of* ω *and* $\boldsymbol{\mu}_0$.

Corollary 1.2. *For each* $(\omega, \kappa_0, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0) \in \Omega_0 \times \mathbb{C} \times \mathbb{C}^{N-1} \times \mathbf{M}_0$, *there exists a small positive number* $r'_0 = r'_0(\omega, \boldsymbol{\kappa}_1, \boldsymbol{\mu}_0)$ *such that system (1.1) admits a solution* $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ *satisfying*

$$\lambda_1 = \mathrm{e}^{-\kappa_0} t^{\omega} \big(1 + O(|t| + |\mathrm{e}^{-\kappa_0} t^{\omega}| + |\mathrm{e}^{\kappa_0} t^{1-\omega}|) \big),$$
$$\lambda_l = \mu_{0,l} + \mathrm{e}^{-\kappa_0} t^{\omega} \big(\kappa_{1,l} + O(|t| + |\mathrm{e}^{-\kappa_0} t^{\omega}| + |\mathrm{e}^{\kappa_0} t^{1-\omega}|) \big) \quad (2 \leqslant l \leqslant N)$$

as $t \to 0$ *through the domain*

$$\Delta'_0(\omega, \kappa_0, r'_0) := \big\{ t \in \mathcal{R}_0 \mid |t| < r'_0, \chi(t) \log |t| < \mathrm{Re}\,\kappa_0 + \log((r'_0)^{1/2}), (1 - \chi(t)) \log |t| < -\mathrm{Re}\,\kappa_0 + \log((r'_0)^{1/2}) \big\},$$

where $\chi(t) = \mathrm{Re}\,\omega - \mathrm{Im}\,\omega (\log |t|)^{-1} \arg t$.

Remark 1.3. It is easy to see that, for a sufficiently small $r^*_0 (< r'_0)$,

$$\Delta^*(\omega, r^*_0) := \{ t \in \mathcal{R}_0 \mid |t| < r^*_0, 0 < \chi(t) < 1 \} \subset \Delta'_0(\omega, \kappa_0, r'_0),$$

and that $|t^{\omega}| = |t|^{\chi(t)}$, $|t^{1-\omega}| = |t|^{1-\chi(t)}$ in $\Delta^*(\omega, r^*_0)$. If $0 < \mathrm{Re}\,\omega < 1$, then, for every large positive number R_0, there exists a small positive number $\tilde{r}_0(R_0)$ such that $\{ t \in \mathcal{R}_0 \mid |t| < \tilde{r}_0(R_0), |\arg t| < R_0 \} \subset \Delta^*(\omega, r^*_0) \subset \Delta'_0(\omega, \kappa_0, r'_0)$. If $\mathrm{Re}\,\omega < 0$ (respectively, $\mathrm{Re}\,\omega > 1$), and if $\mathrm{Im}\,\omega \neq 0$, then $\Delta^*(\omega, r^*_0)$ is a spiral domain.

Remark 1.4. By the symmetry of (1.1) with respect to λ_j $(1 \leqslant j \leqslant N)$, for each l, $2 \leqslant l \leqslant N$ as well, there exists a solution $\big(\lambda^{(l)}_1, \ldots, \lambda^{(l)}_N\big)$ such that

$$\lambda^{(l)}_l = \mathrm{e}^{-\kappa_0} t^{\omega} \big(1 + O(|t| + |\mathrm{e}^{-\kappa_0} t^{\omega}| + |\mathrm{e}^{\kappa_0} t^{1-\omega}|) \big),$$
$$\lambda^{(l)}_j = \mu_{0,j} + \mathrm{e}^{-\kappa_0} t^{\omega} \big(\kappa_{1,j} + O(|t| + |\mathrm{e}^{-\kappa_0} t^{\omega}| + |\mathrm{e}^{\kappa_0} t^{1-\omega}|) \big) \quad (j \neq l)$$

as $t \to 0$ through $\Delta'_0(\omega, \kappa_0, r'_0)$.

Our main result is proved in section 3 by the use of preparatory lemmas given in section 2. For the simplicity of description, we show it in the case where $N = 2$; and the general case is treated by the same argument. The formal series of solutions are constructed by iteration, and their convergence is proved by using a kind of majorant series.

2. Preliminaries

In what follows we consider the case where $N = 2$. Let Ω, K $(:= \mathbf{K})$, $M(:= \mathbf{M})$ be the domains in $\mathbb{C}$ satisfying the suppositions of theorem 1.1, and suppose that $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$. We use the notation below.

(1) $\mathfrak{R}$ denotes the set of formal series expressed as

$$\phi = \sum_{p\geqslant 1} \gamma_p^0(\omega, \kappa_1, \mu_0) t^p + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \gamma_{pq}^1(\omega, \kappa_1, \mu_0) t^p (\mathrm{e}^{-\kappa_0} t^\omega)^q + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \gamma_{pq}^2(\omega, \kappa_1, \mu_0) t^p (\mathrm{e}^{\kappa_0} t^{1-\omega})^q, \tag{2.1}$$

where $\gamma_p^0, \gamma_{pq}^1, \gamma_{pq}^2 \in \mathbb{C}(\omega, \mu_0)[\kappa_1]$.

(2) For $\phi \in \mathfrak{R}$ expressed as (2.1), we define $\|\phi\| = \|\phi(t)\| = \|\phi\|(|t|)$ by

$$\|\phi\| = \sum_{p\geqslant 1} \left|\gamma_p^0(\omega, \kappa_1, \mu_0)\right| |t|^p + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \left(\left|\gamma_{pq}^1(\omega, \kappa_1, \mu_0)\right| + \left|\gamma_{pq}^2(\omega, \kappa_1, \mu_0)\right|\right) |t|^{p+q/2},$$

which is a function of $(\omega, \kappa_1, \mu_0, |t|) \in \Omega \times K \times M \times \{\tau | \tau \geqslant 0\}$, not necessarily finite valued.

(3) We set

$$\mathfrak{R}(\Omega, K, M, r) := \{\phi \in \mathfrak{R} \mid \sup\{\|\phi\|(r) \mid (\omega, \kappa_1, \mu_0) \in \Omega \times K \times M\} < \infty\}.$$

(4) For $\phi \in \mathfrak{R}$ expressed as (2.1), we define the operators $\mathcal{I}_0 : \mathfrak{R} \to \mathfrak{R}$ and $\mathcal{I}_\omega : \mathfrak{R} \to \mathfrak{R}$ by

$$\mathcal{I}_0[\phi] := \sum_{p\geqslant 1} \frac{\gamma_p^0(\omega, \kappa_1, \mu_0)}{p} t^p + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \frac{\gamma_{pq}^1(\omega, \kappa_1, \mu_0)}{p + \omega q} t^p (\mathrm{e}^{-\kappa_0} t^\omega)^q + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \frac{\gamma_{pq}^2(\omega, \kappa_1, \mu_0)}{p + (1-\omega) q} t^p (\mathrm{e}^{\kappa_0} t^{1-\omega})^q,$$

and

$$\mathcal{I}_\omega[\phi] := \sum_{p\geqslant 1} \frac{\gamma_p^0(\omega, \kappa_1, \mu_0)}{p+\omega} t^p + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \frac{\gamma_{pq}^1(\omega, \kappa_1, \mu_0)}{p + \omega q + \omega} t^p (\mathrm{e}^{-\kappa_0} t^\omega)^q + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}} \frac{\gamma_{pq}^2(\omega, \kappa_1, \mu_0)}{p + (1-\omega) q + \omega} t^p (\mathrm{e}^{\kappa_0} t^{1-\omega})^q.$$

Proposition 2.1. *(a) $\mathfrak{R}$ is a ring.*

(b) Suppose that $\phi \in \mathfrak{R}(\Omega, K, M, r)$. Then, ϕ is a holomorphic function in the domain $\Delta_0(\Omega, K, M, r)$ (cf theorem 1.1), and satisfies $\|\phi\| = O(|t|^{1/2})$ uniformly for $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$. Moreover, the series ϕ can be differentiated term by term with respect to each variable.

Proof. The assertion (a) follows from the fact that, for every pair $(p_0, q_0) \in (\mathbb{N} \cup \{0\})^2$, the number of triples (l_0, l_1, l_2) satisfying $t^{l_0}(\mathrm{e}^{-\kappa_0} t^\omega)^{l_1} (\mathrm{e}^{\kappa_0} t^{1-\omega})^{l_2} = t^{p_0}(\mathrm{e}^{-\kappa_0} t^\omega)^{q_0}$ or $=$

$t^{p_0}(\mathrm{e}^{\kappa_0}t^{1-\omega})^{q_0}$ is finite. Observe that, for any $r' < r$, the series $\phi \in \mathfrak{R}(\Omega, K, M, r)$ converges uniformly and absolutely in the domain $\Delta_0(\Omega, K, M, r')$; which implies the assertion (b). □

The following inequalities are easily checked.

Proposition 2.2. *Suppose that* $\phi, \psi \in \mathfrak{R}(\Omega, K, M, r)$.

(a) If $\|\phi\| \equiv 0$ *for* $|t| < r$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$, *then* $\phi \equiv 0$.

(b) If $\gamma_0 \in \mathbb{C}(\omega, \mu_0)[\kappa_1]$, *then* $\|\gamma_0\phi\| = |\gamma_0|\|\phi\|$ *for* $|t| < r$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$.

(c) $\|\phi + \psi\| \leqslant \|\phi\| + \|\psi\|$ *for* $|t| < r$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$.

(d) $\|\phi\psi\| \leqslant \|\phi\|\|\psi\|$ *for* $|t| < r$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$.

Suppose that the power series

$$f(y_1, \ldots, y_h) = \sum_{|\mathbf{k}|\geqslant 1} c_{\mathbf{k}} y_1^{k_1} \cdots y_h^{k_h}, \qquad c_{\mathbf{k}} \in \mathbb{C}(\omega, \mu_0)[\kappa_1],$$

$$\mathbf{k} = (k_1, \ldots, k_h), \qquad |\mathbf{k}| = k_1 + \cdots + k_h$$

converges for $|y_j| < \rho_j$ $(1 \leqslant j \leqslant h)$. Then, by the same argument as in the proof of [17, proposition 3.3], we have

Proposition 2.3. *(a) If* $\phi_j \in \mathfrak{R}$ $(1 \leqslant j \leqslant h)$, *then* $f(\phi_1, \ldots, \phi_h) \in \mathfrak{R}$.

(b) If $\|\phi_j\| < \rho_j$ $(1 \leqslant j \leqslant h)$ *for* $|t| < r$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$, *then*

$$\|f(\phi_1, \ldots, \phi_h)\| \leqslant \sum_{|\mathbf{k}|\geqslant 1} |c_{\mathbf{k}}|\|\phi_1\|^{k_1} \cdots \|\phi_h\|^{k_h}.$$

Clearly, if $\phi \in \mathfrak{R}$, then $\mathcal{I}_0[\phi] \in \mathfrak{R}$ and $\mathcal{I}_\omega[\phi] \in \mathfrak{R}$. The following fact indicates that $\mathcal{I}_0$ and $\mathcal{I}_\omega$ correspond to certain kinds of formal integral operators.

Proposition 2.4. *If* $\mathcal{I}_0[\phi] \in \mathfrak{R}(\Omega, K, M, r)$ *(respectively,* $\mathcal{I}_\omega[\phi] \in \mathfrak{R}(\Omega, K, M, r)$*), then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_0[\phi] = t^{-1}\phi \qquad \left(\textit{respectively, } \frac{\mathrm{d}}{\mathrm{d}t}(t^\omega \mathcal{I}_\omega[\phi]) = t^{\omega-1}\phi\right)$$

for $(\omega, \kappa_0, \kappa_1, \mu_0, t) \in \Omega \times \mathbb{C} \times K \times M \times \mathcal{R}_0$, $|t| < r$, $|\mathrm{e}^{-\kappa_0}t^\omega| < r^{1/2}$, $|\mathrm{e}^{\kappa_0}t^{1-\omega}| < r^{1/2}$.

Since $\mathrm{cl}(\Omega) \subset \Omega_0$, there exists a small positive number $\varepsilon_0 (< 1/2)$ such that

$$\Omega \subset \{\omega \mid \varepsilon_0 < \mathrm{Re}\,\omega < 1 - \varepsilon_0\} \cup \{\omega \mid |\mathrm{Im}\,\omega| > \varepsilon_0|\mathrm{Re}\,\omega - 1/2|\}. \tag{2.2}$$

Proposition 2.5. *If* $\phi \in \mathfrak{R}(\Omega, K, M, r)$, *then*

$$\|\mathcal{I}_0[\phi]\| \leqslant \varepsilon_0^{-1}\int_0^{|t|} \tau^{-1}\|\phi\|(\tau)\,\mathrm{d}\tau \quad \textit{and} \quad \|\mathcal{I}_\omega[\phi]\| \leqslant \varepsilon_0^{-1}|t|^{-1/2}\int_0^{|t|} \tau^{-1/2}\|\phi\|(\tau)\,\mathrm{d}\tau$$

for $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$, $|t| < r$.

To prove this proposition, we note the following:

Lemma 2.6. *For every* $(p, q) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$ *and for every* $\omega \in \Omega$,

$$\frac{p + q/2}{|p + \omega q|} \leqslant \varepsilon_0^{-1}, \qquad \frac{p + q/2}{|p + (1 - \omega)q|} \leqslant \varepsilon_0^{-1}.$$

Proof. By (2.2), if $\omega \in \Omega$, then either $\varepsilon_0 < \mathrm{Re}\,\omega < 1 - \varepsilon_0$ or $\varepsilon_0|\mathrm{Re}\,\omega - 1/2| < |\mathrm{Im}\,\omega|$ holds. If $\varepsilon_0 < \mathrm{Re}\,\omega < 1 - \varepsilon_0$, then, for every $(p, q) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$,

$$\max\left\{\frac{p + q/2}{|p + \omega q|}, \frac{p + q/2}{|p + (1 - \omega)q|}\right\} \leqslant \frac{p + q/2}{p + \varepsilon_0 q} \leqslant \frac{1}{2\varepsilon_0}.$$

If $\varepsilon_0|\mathrm{Re}\,\omega - 1/2| < |\mathrm{Im}\,\omega|$, then, for every $a \in \mathbb{R}\backslash\{0\}$, we have $\varepsilon_0|\mathrm{Re}(a(\omega - 1/2))| < |\mathrm{Im}(a(\omega - 1/2))|$, and, hence

$$\frac{p + q/2}{|p + \omega q|} = \left|1 + \frac{\omega - 1/2}{p/q + 1/2}\right|^{-1} \leqslant \left(\frac{3\varepsilon_0}{2\sqrt{1+\varepsilon_0^2}}\right)^{-1} < \varepsilon_0^{-1},$$

$$\frac{p + q/2}{|p + (1-\omega) q|} = \left|1 + \frac{1/2 - \omega}{p/q + 1/2}\right|^{-1} \leqslant \left(\frac{3\varepsilon_0}{2\sqrt{1+\varepsilon_0^2}}\right)^{-1} < \varepsilon_0^{-1}$$

for every $(p, q) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$. This completes the proof of the lemma. □

Proof of proposition 2.5. By lemma 2.6,

$$\frac{|t|^p}{|p+\omega|} \leqslant \varepsilon_0^{-1}|t|^{-1/2}\int_0^{|t|}\tau^{-1/2}\tau^p\,\mathrm{d}\tau \qquad (p \geqslant 1),$$
$$\frac{|t|^{p+q/2}}{|p+\omega q+\omega|} \leqslant \varepsilon_0^{-1}|t|^{-1/2}\int_0^{|t|}\tau^{-1/2}\tau^{p+q/2}\,\mathrm{d}\tau \qquad (p \geqslant 0, q \geqslant 1),$$
$$\frac{|t|^{p+q/2}}{|p+(1-\omega) q+\omega|} \leqslant \varepsilon_0^{-1}|t|^{-1/2}\int_0^{|t|}\tau^{-1/2}\tau^{p+q/2}\,\mathrm{d}\tau \qquad (p \geqslant 0, q \geqslant 1)$$

for $\omega \in \Omega$. Then, for $\phi \in \mathfrak{R}(\Omega, K, M, r)$ expressed as (2.1), we have

$$\|\mathcal{I}_\omega[\phi]\| \leqslant \varepsilon_0^{-1}|t|^{-1/2}\int_0^{|t|}\tau^{-1/2}\Bigg(\sum_{p\geqslant 1}|\gamma_p^0|\tau^p + \sum_{\substack{p\geqslant 0\\ q\geqslant 1}}\left(|\gamma_{pq}^1| + |\gamma_{pq}^2|\right)\tau^{p+q/2}\Bigg)\,\mathrm{d}\tau$$
$$= \varepsilon_0^{-1}|t|^{-1/2}\int_0^{|t|}\tau^{-1/2}\|\phi\|(\tau)\,\mathrm{d}\tau$$

for $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$, $|t| < r$, which is the second inequality. The first inequality is verified by the same argument ([17, proposition 3.5]). □

3. Proof of theorem 1.1

3.1. Transformation and a system of equations

Suppose that $N = 2$. Setting $\lambda := \lambda_1$, $\mu := \lambda_2$, we write (1.1) in the form

$$\lambda'' = \Phi_2(t, \lambda, \mu, \lambda', \mu'), \qquad \mu'' = \Phi_2(t, \mu, \lambda, \mu', \lambda')$$

with $\Phi_2(t, \lambda, \mu, \tilde{\lambda}, \tilde{\mu})$ given by (1.2). The suppositions in theorem 1.1 are written as

$$(\omega, \kappa_0, \kappa_1, \mu_0) \in \Omega \times \mathbb{C} \times K \times M, \qquad \mathrm{cl}(\Omega) \subset \Omega_0,$$
$$\mathrm{cl}(M) \subset M_0 = \mathbb{C} \setminus \{0, 1, s_0\}, \qquad s_0 \in \mathbb{C} \setminus \{0, 1\}.$$

By $\lambda = \mathrm{e}^{-w}$, $\mu = \mu_0 + z$, system (1.1) is changed into

$$t(tw')' = F_{20}(\mathrm{e}^{-w}, t\,\mathrm{e}^{w}, z)(tw')^2 + F_{11}(t, \mathrm{e}^{-w}, z)(tw')(tz')$$
$$+ F_{02}(t, \mathrm{e}^{-w}, t\,\mathrm{e}^{w}, z)\,\mathrm{e}^{w}(tz')^2 + F_{10}(t, t\,\mathrm{e}^{w}, z)(tw') + F_{00}(t, \mathrm{e}^{-w}, t\,\mathrm{e}^{w}, z),$$
$$t(tz')' + (tw')(tz') = G_{02}(t, \mathrm{e}^{-w}, z)(tz')^2 + G_{11}(\mathrm{e}^{-w}, t\,\mathrm{e}^{w}, z)(tw')(tz')$$
$$+ G_{20}(t, \mathrm{e}^{-w}, t\,\mathrm{e}^{w}, z)\,\mathrm{e}^{-2w}(tw')^2 + G_{01}(t, t\,\mathrm{e}^{w}, z)(tz')$$
$$+ G_{00}(t, \mathrm{e}^{-w}, t\,\mathrm{e}^{w}, z)\,\mathrm{e}^{-w},$$

where

$$F_{20}(\xi,\eta,z)=-\frac{1}{2}\left(\frac{\eta}{1-\eta}+\frac{\xi}{\xi-1}+\frac{\xi}{\xi-s_0}-\frac{\xi}{\xi-\mu_0-z}\right),$$

$$F_{11}(t,\xi,z)=-\frac{\xi-t}{(\mu_0+z-t)(\mu_0+z-\xi)},$$

$$F_{02}(t,\xi,\eta,z)=-\frac{(\xi-1)(\xi-s_0)(\mu_0+z-t)}{2(\mu_0+z)(\mu_0-1+z)(\mu_0-s_0+z)(\mu_0+z-\xi)(1-\eta)},$$

$$F_{10}(t,\eta,z)=-\left(\frac{t}{t-1}+\frac{t}{t-s_0}-\frac{\eta}{\eta-1}-\frac{t}{t-\mu_0-z}\right),$$

$$F_{00}(t,\xi,\eta,z)=-\frac{2(\xi-1)(1-\eta)(\xi-s_0)(\mu_0+z-t)^2}{(t-1)^2(t-s_0)^2(\xi-\mu_0-z)}\left(a_\infty\xi-\frac{a_0s_0\eta}{\mu_0+z}\right.$$
$$\left.+\frac{a_1(s_0-1)(t-1)\xi}{(\mu_0+z-1)(\xi-1)^2}+\frac{a_2(t-1)(t-s_0)\eta}{(\mu_0+z-t)(1-\eta)^2}+\frac{a_3s_0(s_0-1)(s_0-t)\xi}{(\mu_0-s_0+z)(\xi-s_0)^2}\right),$$

$$G_{02}(t,\xi,z)=\frac{1}{2}\left(\frac{1}{\mu_0+z}+\frac{1}{\mu_0-1+z}+\frac{1}{\mu_0-t+z}+\frac{1}{\mu_0-s_0+z}-\frac{1}{\mu_0-\xi+z}\right),$$

$$G_{11}(\xi,\eta,z)=\frac{\xi}{\xi-\mu_0-z}-\frac{\eta}{1-\eta},$$

$$G_{20}(t,\xi,\eta,z)=\frac{(\mu_0+z)(\mu_0-1+z)(\mu_0-s_0+z)(1-\eta)}{2(\xi-1)(\xi-s_0)(\mu_0-t+z)(\xi-\mu_0-z)},$$

$$G_{01}(t,\eta,z)=-\left(\frac{t}{t-1}+\frac{t}{t-s_0}-\frac{\eta}{\eta-1}-\frac{t}{t-\mu_0-z}\right),$$

$$G_{00}(t,\xi,\eta,z)=\frac{2(\mu_0+z)(\mu_0-1+z)(\mu_0-t+z)(\mu_0-s_0+z)(1-\eta)^2}{(t-1)^2(t-s_0)^2(\mu_0-\xi+z)}$$
$$\times\left(a_\infty\xi-\frac{a_0s_0t}{(\mu_0+z)^2}+\frac{a_1(s_0-1)(t-1)\xi}{(\mu_0-1+z)^2(\xi-1)}\right.$$
$$\left.+\frac{a_2t(t-1)(t-s_0)}{(\mu_0-t+z)^2(1-\eta)}+\frac{a_3s_0(s_0-1)(s_0-t)\xi}{(\mu_0-s_0+z)^2(\xi-s_0)}\right).$$

Let us make the further change of variables $w=-\omega\log t+\kappa_0+u$, $z=\mathrm{e}^{-\kappa_0}t^{\omega}(\kappa_1+v)$. Then, we note the following relations:

$$\mathrm{e}^{-w}=\mathrm{e}^{-\kappa_0}t^{\omega}\,\mathrm{e}^{-u},\qquad t\,\mathrm{e}^{w}=\mathrm{e}^{\kappa_0}t^{1-\omega}\,\mathrm{e}^{u},\qquad tw'=tu'-\omega,\qquad t(tw')'=t(tu')',$$
$$tz'=\mathrm{e}^{-\kappa_0}t^{\omega}(\omega(\kappa_1+v)+tv'),\qquad t(tz')'=\mathrm{e}^{-\kappa_0}t^{\omega}(\omega^2(\kappa_1+v)+2\omega tv'+t(tv')').$$

Using them, and observing that $\mathrm{e}^{w}(tz')^2=\mathrm{e}^{-\kappa_0}t^{\omega}\,\mathrm{e}^{u}(\omega(\kappa_1+v)+tv')^2$, we obtain the system of equations

$$t(tu')'=\Psi_1(t,\mathrm{e}^{-\kappa_0}t^{\omega},\mathrm{e}^{\kappa_0}t^{1-\omega})+H_1(t,\mathrm{e}^{-\kappa_0}t^{\omega},\mathrm{e}^{\kappa_0}t^{1-\omega},u,v,tu',tv'),$$
$$t(tv')'+\omega tv'=-(\omega v+tv')tu'-\omega\kappa_1tu'+\Psi_2(t,\mathrm{e}^{-\kappa_0}t^{\omega},\mathrm{e}^{\kappa_0}t^{1-\omega})\qquad(3.1)$$
$$+H_2(t,\mathrm{e}^{-\kappa_0}t^{\omega},\mathrm{e}^{\kappa_0}t^{1-\omega},u,v,tu',tv').$$

Here

$$\Psi_1(t,\xi,\eta)=\omega^2F_{20}(\xi,\eta,\kappa_1\xi)-\omega^2\kappa_1\xi F_{11}(t,\xi,\kappa_1\xi)+\omega^2\kappa_1^2\xi F_{02}(t,\xi,\eta,\kappa_1\xi)$$
$$-\omega F_{10}(t,\eta,\kappa_1\xi)+F_{00}(t,\xi,\eta,\kappa_1\xi),$$

$$\Psi_2(t,\xi,\eta)=\omega^2\kappa_1^2\xi G_{02}(t,\xi,\kappa_1\xi)-\omega^2\kappa_1G_{11}(\xi,\eta,\kappa_1\xi)+\omega^2\xi G_{20}(t,\xi,\eta,\kappa_1\xi)$$
$$+\omega\kappa_1G_{01}(t,\eta,\kappa_1\xi)+G_{00}(t,\xi,\eta,\kappa_1\xi),$$

which satisfy $\Psi_h(0,0,0) = 0$ $(h = 1, 2)$; and $H_h(t, \xi, \eta, u, v, \tilde{u}, \tilde{v})$ $(h = 1, 2)$ are rational functions of $t, \xi, \eta, \mathrm{e}^{\pm u}, v, \tilde{u}, \tilde{v}$ with the properties

$$H_h(0,0,0,u,v,\tilde{u},\tilde{v}) \equiv 0, \qquad H_h(t,\xi,\eta,0,0,0,0) \equiv 0.$$

By the condition $\mu_0 \in \mathbb{C}\backslash\{0, 1, s_0\}$, $s_0 \neq 0, 1$, these functions are expanded into power series around the origin, whose coefficients belong to $\mathbb{C}(\mu_0)[\omega, \kappa_1]$. The transformation and its resultant system are summarized as follows:

Proposition 3.1. *For any* $(\omega, \kappa_0, \kappa_1, \mu_0) \in \Omega \times \mathbb{C} \times K \times M$, *by the transformation*

$$\lambda = \mathrm{e}^{-\kappa_0} t^{\omega} \mathrm{e}^{-u}, \qquad \mu = \mu_0 + \mathrm{e}^{-\kappa_0} t^{\omega} (\kappa_1 + v), \tag{3.2}$$

system (1.1) is changed into (3.1), whose right-hand members have the following properties:

(a) $\Psi_h(t, \xi, \eta)$ $(h = 1, 2)$ *are holomorphic for* $|t| < r_1$, $|\xi| < r_1^{1/2}$, $|\eta| < r_1^{1/2}$, *and are expanded into the convergent power series*

$$\Psi_h(t,\xi,\eta) = \sum_{p_0+p_1+p_2 \geqslant 1} b^{(h)}_{p_0,p_1,p_2}(\omega, \kappa_1, \mu_0) t^{p_0} \xi^{p_1} \eta^{p_2}$$

with $b^{(h)}_{p_0,p_1,p_2}(\omega, \kappa_1, \mu_0) \in \mathbb{C}(\mu_0)[\omega, \kappa_1]$, *where* $r_1 = r_1(\Omega, K, M)$ *is a sufficiently small positive number;*

(b) $H_h(t, \xi, \eta, u, v, \tilde{u}, \tilde{v})$ $(h = 1, 2)$ *are holomorphic for* $|t| < r_1$, $|\xi| < r_1^{1/2}$, $|\eta| < r_1^{1/2}$, $|u| < \rho_0$, $|v| < \rho_0$, $|\tilde{u}| < \infty$, $|\tilde{v}| < \infty$, *and are expanded into the convergent power series*

$$H_h(t,\xi,\eta,u,v,\tilde{u},\tilde{v}) = \sum_{\substack{l_1+l_2+l_3+l_4 \geqslant 1 \\ 0 \leqslant l_3+l_4 \leqslant 2}} \left(\sum_{p_0+p_1+p_2 \geqslant 1} c^{(h,l_1,l_2,l_3,l_4)}_{p_0,p_1,p_2}(\omega, \kappa_1, \mu_0) t^{p_0} \xi^{p_1} \eta^{p_2} \right) u^{l_1} v^{l_2} \tilde{u}^{l_3} \tilde{v}^{l_4}$$

with $c^{(h,l_1,l_2,l_3,l_4)}_{p_0,p_1,p_2}(\omega, \kappa_1, \mu_0) \in \mathbb{C}(\mu_0)[\omega, \kappa_1]$, *where* $\rho_0 = \rho_0(\Omega, K, M)$ *is a sufficiently small positive number.*

In what follows r_1 and ρ_0 denote the positive constants given above. By propositions 2.1(b), 2.3 and 3.1, we immediately have the following:

Proposition 3.2. *Let* r' *be an arbitrary fixed positive number such that* $r' < r_1/2$. *Suppose that* $\phi_m, \psi_m, \tilde{\phi}_m, \tilde{\psi}_m \in \mathfrak{R}(\Omega, K, M, r')$ $(m = 1, 2)$ *satisfy* $\|\phi_m\| < \rho_0/2$, $\|\psi_m\| < \rho_0/2$, $\|\tilde{\phi}_m\| < \rho_0/2$, $\|\tilde{\psi}_m\| < \rho_0/2$ *for* $|t| < r'$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$. *Then,*

$$H_h(t, \mathrm{e}^{-\kappa_0} t^{\omega}, \mathrm{e}^{\kappa_0} t^{1-\omega}, \phi_m, \psi_m, \tilde{\phi}_m, \tilde{\psi}_m) \in \mathfrak{R}(\Omega, K, M, r') \quad (h = 1, 2),$$

and

$$\begin{aligned} &\|H_h(t, \mathrm{e}^{-\kappa_0} t^{\omega}, \mathrm{e}^{\kappa_0} t^{1-\omega}, \phi_2, \psi_2, \tilde{\phi}_2, \tilde{\psi}_2) - H_h(t, \mathrm{e}^{-\kappa_0} t^{\omega}, \mathrm{e}^{\kappa_0} t^{1-\omega}, \phi_1, \psi_1, \tilde{\phi}_1, \tilde{\psi}_1)\| \\ &\qquad \leqslant L_0 |t|^{1/2} (\|\phi_2 - \phi_1\| + \|\psi_2 - \psi_1\| + \|\tilde{\phi}_2 - \tilde{\phi}_1\| + \|\tilde{\psi}_2 - \tilde{\psi}_1\|), \end{aligned}$$

uniformly for $|t| < r'$, $(\omega, \kappa_0, \kappa_1, \mu_0) \in \Omega \times \mathbb{C} \times K \times M$, *where* L_0 *is some positive constant independent of* r'.

Set $g(v, \tilde{u}, \tilde{v}) := (\omega v + \tilde{v})\tilde{u} + \omega \kappa_1 \tilde{u}$. Since $|t|^{-1/2}\|\tilde{\phi}_m\|(|t|)$ is monotonically increasing with respect to $|t|$, it follows that, under the conditions of proposition 3.2, $\|\tilde{\phi}_m\|(|t|) \leqslant (\rho_0/2)(r')^{-1/2}|t|^{1/2}$ uniformly for $|t| < r'$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$. Using this fact, we have

Proposition 3.3. *Under the same suppositions as above, there exists a positive constant* L_1 *independent of* r' *such that*

$$\begin{aligned} &\|g(\psi_2, \tilde{\phi}_2, \tilde{\psi}_2) - g(\psi_1, \tilde{\phi}_1, \tilde{\psi}_1)\| \\ &\qquad \leqslant L_1 (\|\tilde{\phi}_2 - \tilde{\phi}_1\| + (r')^{-1/2} |t|^{1/2} (\|\psi_2 - \psi_1\| + \|\tilde{\psi}_2 - \tilde{\psi}_1\|)) \end{aligned}$$

uniformly for $|t| < r'$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$.

Proposition 3.4. *For an arbitrary fixed positive number satisfying $r' < r_1/2$, we have*

$$\|\Psi_h(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega})\| \leqslant L_2|t|^{1/2}$$

uniformly for $|t| < r'$, $(\omega, \kappa_0, \kappa_1, \mu_0) \in \Omega \times \mathbb{C} \times K \times M$, where L_2 is some positive constant independent of r'.

3.2. Construction of an iterative sequence

Note that (3.1) is written in the form

$$\begin{aligned}
&\tilde{u}' = t^{-1}\Psi_1(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}) + t^{-1}H_1(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}, u, v, \tilde{u}, \tilde{v}),\\
&u' = t^{-1}\tilde{u},\\
&(t^{\omega}\tilde{v})' = -t^{\omega-1}g(v, \tilde{u}, \tilde{v}) + t^{\omega-1}\Psi_2(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega})\\
&\qquad\qquad + t^{\omega-1}H_2(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}, u, v, \tilde{u}, \tilde{v}),\\
&v' = t^{-1}\tilde{v}.
\end{aligned} \tag{3.3}$$

In view of this together with proposition 2.4, we consider the corresponding system of formal integral equations for $u, v, \tilde{u}, \tilde{v} \in \mathfrak{R}$:

$$\begin{aligned}
&\tilde{u} = \mathcal{I}_0[\Psi_1(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega})] + \mathcal{I}_0[H_1(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}, u, v, \tilde{u}, \tilde{v})],\\
&u = \mathcal{I}_0[\tilde{u}],\\
&\tilde{v} = -\mathcal{I}_{\omega}[g(v, \tilde{u}, \tilde{v})] + \mathcal{I}_{\omega}[\Psi_2(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega})]\\
&\qquad + \mathcal{I}_{\omega}[H_2(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}, u, v, \tilde{u}, \tilde{v})],\\
&v = \mathcal{I}_0[\tilde{v}].
\end{aligned} \tag{3.4}$$

To construct a solution of (3.4), we define the sequence $u_{\nu}(t), v_{\nu}(t), \tilde{u}_{\nu}(t), \tilde{v}_{\nu}(t) \in \mathfrak{R}$ by

$$\begin{aligned}
&u_0(t) = v_0(t) = \tilde{u}_0(t) = \tilde{v}_0(t) \equiv 0,\\
&\tilde{u}_{\nu}(t) = \mathcal{I}_0[\Psi_1(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega})]\\
&\qquad + \mathcal{I}_0[H_1(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}, u_{\nu-1}(t), v_{\nu-1}(t), \tilde{u}_{\nu-1}(t), \tilde{v}_{\nu-1}(t))],\\
&u_{\nu}(t) = \mathcal{I}_0[\tilde{u}_{\nu}(t)],\\
&\tilde{v}_{\nu}(t) = -\mathcal{I}_{\omega}[g(v_{\nu-1}(t), \tilde{u}_{\nu}(t), \tilde{v}_{\nu-1}(t))] + \mathcal{I}_{\omega}[\Psi_2(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega})]\\
&\qquad + \mathcal{I}_{\omega}[H_2(t, \mathrm{e}^{-\kappa_0}t^{\omega}, \mathrm{e}^{\kappa_0}t^{1-\omega}, u_{\nu}(t), v_{\nu-1}(t), \tilde{u}_{\nu}(t), \tilde{v}_{\nu-1}(t))],\\
&v_{\nu}(t) = \mathcal{I}_0[\tilde{v}_{\nu}(t)]
\end{aligned} \tag{3.5}$$

for $\nu \geqslant 1$, and put

$$\begin{aligned}
&U_{\nu}(t) = u_{\nu}(t) - u_{\nu-1}(t), \qquad V_{\nu}(t) = v_{\nu}(t) - v_{\nu-1}(t),\\
&\tilde{U}_{\nu}(t) = \tilde{u}_{\nu}(t) - \tilde{u}_{\nu-1}(t), \qquad \tilde{V}_{\nu}(t) = \tilde{v}_{\nu}(t) - \tilde{v}_{\nu-1}(t).
\end{aligned} \tag{3.6}$$

Then we have

Proposition 3.5. *There exists a positive number $r_0 = r_0(\Omega, K, M)$ such that the estimates*

$$\max\{\|u_{\nu}(t)\|, \|v_{\nu}(t)\|, \|\tilde{u}_{\nu}(t)\|, \|\tilde{v}_{\nu}(t)\|\} < \rho_0/3 \quad (\nu \geqslant 0), \tag{3.7}$$

$$\max\{\|U_{\nu}(t)\|, \|V_{\nu}(t)\|, \|\tilde{U}_{\nu}(t)\|, \|\tilde{V}_{\nu}(t)\|\} \leqslant C_{\nu}|t|^{\nu/2} \quad (\nu \geqslant 1), \tag{3.8}$$

$$\sum_{\nu\geqslant 1} C_{\nu}|t|^{\nu/2} < \rho_0/4 \tag{3.9}$$

are valid for $|t| < r_0$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$, *where*

$$C_\nu = \left(96\varepsilon_0^{-4} L_*\right)^\nu r_0^{-(\nu-1)/2} (\nu!)^{-1}, \qquad L_* = (L_0 + L_1 + 1)(L_0 + L_2 + 1).$$

Proof. Take a number $r_0 < r_1/2$ so small that

$$\sum_{\nu \geqslant 1} C_\nu r_0^{\nu/2} \leqslant C_1 r_0^{1/2} \sum_{\nu \geqslant 1} \left(96\varepsilon_0^{-4} L_*\right)^{\nu-1} (\nu!)^{-1} \leqslant C_1 r_0^{1/2} \exp\left(96\varepsilon_0^{-4} L_*\right) < \rho_0/4.$$

Then (3.9) is valid for $|t| < r_0$. We would like to verify (3.7) and (3.8) by induction on ν. By propositions 2.5 and 3.4, we have, for $|t| < r_0$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$,

$$\begin{aligned}\|\tilde{U}_1(t)\| &= \|\tilde{u}_1(t)\| = \|\mathcal{I}_0[\Psi_1(t, \mathrm{e}^{-\kappa_0} t^\omega, \mathrm{e}^{\kappa_0} t^{1-\omega})]\| \leqslant \varepsilon_0^{-1} \int_0^{|t|} \tau^{-1} \|\Psi_1\|(\tau)\, \mathrm{d}\tau \\ &\leqslant \varepsilon_0^{-1} L_2 \int_0^{|t|} \tau^{-1/2}\, \mathrm{d}\tau \leqslant 2\varepsilon_0^{-1} L_2 |t|^{1/2} \leqslant C_1 |t|^{1/2} < \rho_0/3,\end{aligned}$$

so that $\tilde{u}_1(t) \in \mathfrak{R}(\Omega, K, M, r_0)$; and

$$\begin{aligned}\|U_1(t)\| &= \|u_1(t)\| = \|\mathcal{I}_0[\tilde{u}_1(t)]\| \leqslant \varepsilon_0^{-1} \int_0^{|t|} \tau^{-1} \|\tilde{u}_1\|(\tau)\, \mathrm{d}\tau \\ &\leqslant 4\varepsilon_0^{-2} L_2 |t|^{1/2} \leqslant C_1 |t|^{1/2} < \rho_0/3, \qquad u_1(t) \in \mathfrak{R}(\Omega, K, M, r_0).\end{aligned}$$

Using propositions 3.2–3.4 and the estimates for $\|u_1(t)\|$, $\|\tilde{u}_1(t)\|$ above, we have

$$\begin{aligned}\|\tilde{V}_1(t)\| = \|\tilde{v}_1(t)\| &\leqslant \|\mathcal{I}_\omega[g(0, \tilde{u}_1(t), 0)]\| + \|\mathcal{I}_\omega[\Psi_2(t, \mathrm{e}^{-\kappa_0} t^\omega, \mathrm{e}^{\kappa_0} t^{1-\omega})]\| \\ &\quad + \|\mathcal{I}_\omega[H_2(t, \mathrm{e}^{-\kappa_0} t^\omega, \mathrm{e}^{\kappa_0} t^{1-\omega}, u_1(t), 0, \tilde{u}_1(t), 0)]\| \\ &\leqslant \varepsilon_0^{-1} |t|^{-1/2} \int_0^{|t|} \left(L_1 \tau^{-1/2} \|\tilde{u}_1\|(\tau) + L_2 + L_0(\|u_1\|(\tau) + \|\tilde{u}_1\|(\tau))\right) \mathrm{d}\tau \\ &\leqslant 2\varepsilon_0^{-2} L_1 L_2 |t|^{1/2} + \varepsilon_0^{-1} L_2 |t|^{1/2} + 6\varepsilon_0^{-3} L_0 L_2 |t| \\ &\leqslant 6\varepsilon_0^{-3} (L_0 + L_1 + 1) L_2 |t|^{1/2} \leqslant C_1 |t|^{1/2} < \rho_0/3,\end{aligned}$$

$\tilde{v}_1(t) \in \mathfrak{R}(\Omega, K, M, r_0)$, and hence

$$\|V_1(t)\| = \|v_1(t)\| = \|\mathcal{I}_0[\tilde{v}_1(t)]\| \leqslant 12\varepsilon_0^{-4} (L_0 + L_1 + 1) L_2 |t|^{1/2} \leqslant C_1 |t|^{1/2} < \rho_0/3,$$

$v_1(t) \in \mathfrak{R}(\Omega, K, M, r_0)$. This implies that (3.7) and (3.8) are valid for $\nu = 1$. Suppose that (3.7) and (3.8) are valid for $\nu \leqslant n - 1$. By propositions 2.5 and 3.2, we have, for $n \geqslant 1$ and for $|t| < r_0$,

$$\begin{aligned}\|\tilde{U}_n(t)\| &\leqslant \varepsilon_0^{-1} \int_0^{|t|} L_0 \tau^{-1/2} \left(\|\tilde{U}_{n-1}\|(\tau) + \|\tilde{V}_{n-1}\|(\tau) + \|U_{n-1}\|(\tau) + \|V_{n-1}\|(\tau)\right) \mathrm{d}\tau \\ &\leqslant 4\varepsilon_0^{-1} L_0 \int_0^{|t|} C_{n-1} \tau^{(n-2)/2}\, \mathrm{d}\tau = 8\varepsilon_0^{-1} L_0 n^{-1} C_{n-1} |t|^{n/2} \leqslant C_n |t|^{n/2}.\end{aligned}$$

Combining this with (3.8) and (3.9), we have

$$\|\tilde{u}_n(t)\| \leqslant \sum_{\nu=1}^{n} \|\tilde{U}_\nu(t)\| \leqslant \sum_{\nu=1}^{n} C_\nu |t|^{\nu/2} < \rho_0/3$$

for $|t| < r_0$. By the estimate for $\|\tilde{U}_n(t)\|$ above,

$$\begin{aligned}\|U_n(t)\| &\leqslant \varepsilon_0^{-1} \int_0^{|t|} \tau^{-1} \|\tilde{U}_n\|(\tau)\, \mathrm{d}\tau \\ &\leqslant \varepsilon_0^{-1} \int_0^{|t|} 8\varepsilon_0^{-1} L_0 n^{-1} C_{n-1} \tau^{n/2-1}\, \mathrm{d}\tau = 16\varepsilon_0^{-2} L_0 n^{-2} C_{n-1} |t|^{n/2} \leqslant C_n |t|^{n/2},\end{aligned}$$

which implies $\|u_n(t)\| < \rho_0/3$ for $|t| < r_0$. Furthermore, by proposition 3.3,

$$\begin{aligned}
\|\tilde{V}_n(t)\| &\leqslant \varepsilon_0^{-1} L_1 |t|^{-1/2} \int_0^{|t|} \big(\tau^{-1/2}\|\tilde{U}_n\|(\tau) + r_0^{-1/2}(\|\tilde{V}_{n-1}\|(\tau) + \|V_{n-1}\|(\tau))\big)\,\mathrm{d}\tau \\
&\quad + \varepsilon_0^{-1} L_0 |t|^{-1/2} \int_0^{|t|} \big(\|\tilde{V}_{n-1}\|(\tau) + \|V_{n-1}\|(\tau) + \|\tilde{U}_n\|(\tau) + \|U_n\|(\tau)\big)\,\mathrm{d}\tau \\
&\leqslant \varepsilon_0^{-1}(L_0 + L_1) r_0^{-1/2} |t|^{-1/2} \\
&\quad \times \int_0^{|t|} \big(\tau^{-1/2}(\|\tilde{U}_n\|(\tau) + \|U_n\|(\tau)) + \|\tilde{V}_{n-1}\|(\tau) + \|V_{n-1}\|(\tau)\big)\,\mathrm{d}\tau \\
&\leqslant \varepsilon_0^{-1}(L_0 + L_1) r_0^{-1/2} |t|^{-1/2} \int_0^{|t|} \big(24\varepsilon_0^{-2} L_0 n^{-1} C_{n-1} + 2C_{n-1}\big)\tau^{(n-1)/2}\,\mathrm{d}\tau \\
&\leqslant 48\varepsilon_0^{-3}(L_0 + L_1)(L_0 + 1) r_0^{-1/2} n^{-1} C_{n-1} |t|^{n/2} \leqslant C_n |t|^{n/2}, \\
\|V_n(t)\| &\leqslant \varepsilon_0^{-1} \int_0^{|t|} \tau^{-1} \|\tilde{V}_n\|(\tau)\,\mathrm{d}\tau \\
&\leqslant \varepsilon_0^{-1} \int_0^{|t|} 48\varepsilon_0^{-3}(L_0 + L_1)(L_0 + 1) r_0^{-1/2} n^{-1} C_{n-1} \tau^{n/2-1}\,\mathrm{d}\tau \\
&\leqslant 96\varepsilon_0^{-4}(L_0 + L_1)(L_0 + 1) r_0^{-1/2} n^{-2} C_{n-1} |t|^{n/2} \leqslant C_n |t|^{n/2},
\end{aligned}$$

and we have $\|\tilde{v}_n(t)\| < \rho_0/3$, $\|v_n(t)\| < \rho_0/3$ for $|t| < r_0$. These inequalities imply that (3.7) and (3.8) are valid for $\nu = n$. Thus, we obtain the proposition. □

3.3. Completion of the proof of theorem 1.1

By proposition 3.5, in the series expansions of $U_\nu(t)$, $V_\nu(t)$, $\tilde{U}_\nu(t)$, $\tilde{V}_\nu(t) \in \mathfrak{R}(\Omega, K, M, r_0)$, the coefficients of t^p (respectively, $t^p(\mathrm{e}^{-\kappa_0}t^\omega)^q$, $t^p(\mathrm{e}^{\kappa_0}t^{1-\omega})^q$) vanish for p such that $p < \nu/2$ (respectively, for (p, q) such that $p + q/2 < \nu/2$). Consequently, we obtain the series

$$u(t) = \sum_{\nu\geqslant 1} U_\nu(t), \qquad v(t) = \sum_{\nu\geqslant 1} V_\nu(t), \qquad \tilde{u}(t) = \sum_{\nu\geqslant 1} \tilde{U}_\nu(t), \qquad \tilde{v}(t) = \sum_{\nu\geqslant 1} \tilde{V}_\nu(t)$$

belonging to $\mathfrak{R}$. By (3.7), (3.8) and (3.9), the estimates

$$\begin{aligned}
&\max\{\|u(t)\|, \|v(t)\|, \|\tilde{u}(t)\|, \|\tilde{v}(t)\|\} < \rho_0/3, \\
&\max\{\|u(t) - u_{n-1}(t)\|, \|v(t) - v_{n-1}(t)\|, \\
&\|\tilde{u}(t) - \tilde{u}_{n-1}(t)\|, \|\tilde{v}(t) - \tilde{v}_{n-1}(t)\|\} = O(|t|^{n/2})
\end{aligned} \tag{3.10}$$

are valid uniformly for $|t| < r_0$, $(\omega, \kappa_1, \mu_0) \in \Omega \times K \times M$. Using (3.10) combined with propositions 2.5, 3.2 and 3.3, we derive

$$\begin{aligned}
&\|\mathcal{I}_0[H_1(t, \mathrm{e}^{-\kappa_0}t^\omega, \mathrm{e}^{\kappa_0}t^{1-\omega}, u(t), v(t), \tilde{u}(t), \tilde{v}(t))] \\
&\qquad - \mathcal{I}_0[H_1(t, \mathrm{e}^{-\kappa_0}t^\omega, \mathrm{e}^{\kappa_0}t^{1-\omega}, u_{n-1}(t), v_{n-1}(t), \tilde{u}_{n-1}(t), \tilde{v}_{n-1}(t))]\| = O(|t|^{n/2}), \\
&\|\mathcal{I}_\omega[H_2(t, \mathrm{e}^{-\kappa_0}t^\omega, \mathrm{e}^{\kappa_0}t^{1-\omega}, u(t), v(t), \tilde{u}(t), \tilde{v}(t))] \\
&\qquad - \mathcal{I}_\omega[H_2(t, \mathrm{e}^{-\kappa_0}t^\omega, \mathrm{e}^{\kappa_0}t^{1-\omega}, u_n(t), v_{n-1}(t), \tilde{u}_n(t), \tilde{v}_{n-1}(t))]\| = O(|t|^{n/2}), \\
&\|\mathcal{I}_\omega[g(v(t), \tilde{u}(t), \tilde{v}(t))] - \mathcal{I}_\omega[g(v_{n-1}(t), \tilde{u}_n(t), \tilde{v}_{n-1}(t))]\| = O(|t|^{n/2}).
\end{aligned}$$

Therefore, the quadruplet $(u(t), v(t), \tilde{u}(t), \tilde{v}(t)) \in \mathfrak{R}(\Omega, K, M, r_0)^4$ satisfies system (3.4) for $(\omega, \kappa_0, \kappa_1, \mu_0, t) \in \Delta_0(\Omega, K, M, r_0)$. By proposition 2.4, this is also a solution of (3.3). Substitution of this into (3.2) yields the desired solution of (1.1). This completes the proof of theorem 1.1.

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